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FLOW OF VISCOUS FLUID IN SPHERICAL LAYERS

(Review)

I. M. Yavorskaya and N. M. Astaf'yeva

Translation of "Techeniye vyazkoy zhidkosti v sfericheskikh sloyakh (Obzor)," Academy of Sciences of the USSR, Institute of Space Research, Report No. Ob-10, Moscow, 1974, 42 pages.

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16. Abstract

A review is given of works devoted to the study of hydrodynamic motions of a fluid in spherical layers and their stability. The investigation of such flow with various values of physical parameters and various boundary conditions is of significant interest for problems of astro- and geophysics with respect to problems of global circulation in stellar envelopes and large scale flow in planetary atmospheres.

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ABSTRACT. A review is given of works devoted to the study of hydrodynamic motions of a fluid in spherical layers and their stability. The investigation of such flow with various values of physical parameters and various boundary conditions is of significant interest for problems of astro- and geophysics with respect to problems of global circulation in stellar envelopes and large scale flow in planetary atmospheres. /2*

Introduction

Investigation of hydrodynamic motions of fluid in spherical layers with various values of the physical parameters and various boundary conditions is of obvious interest for problems of astro- and geophysics. Actually, meridional circulations in stellar envelopes and large scale flow in planetary atmospheres are described by hydrodynamic equations, taking into account various physical processes occurring in these regions (sources and transfer of energy, ionization, etc.), and certainly taking into account rotation. Along with the calculation of complex specific models, taking into account various physical factors, it is necessary to study simplified models permitting, however, one to obtain very general regularities. There is great interest in this plan in studying the motions of a viscous fluid in spherical layers, which arise as a result of shear, internal, or /3

*Numbers in the margin indicate pagination of original foreign text.

external supply of heat, rotation, etc. Replacement of the spherical layers by plane layers, usually applied for simplification in many problems of geophysics, is far from always being correct and can lead to incorrect qualitative conclusions, to say nothing of the inaccuracy of quantitative evaluations. It is known, for example [1], that instability phenomena in closed and unclosed regions differ significantly and the solutions obtained in the plane layer approximation for supercritical values of the parameters cannot reflect the actual flow pattern. The spherical geometry leads to significant complications, and /4
even in the simplest version of the problem on the axially symmetric motion of an incompressible viscous fluid between two concentric spheres (Figure 1) rotating about a common axis with different speeds, reduces to a complex boundary value problem for the system of nonlinear partial differential equations:

$$u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \vartheta} - \frac{v^2+w^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{Re}\left[\nabla^2 u - \frac{2}{r^2}\left(\frac{\partial v}{\partial \vartheta} + u + v\,ctg\,\vartheta\right)\right], \tag{0.1}$$

$$u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \vartheta} + \frac{uv - w^2 ctg\,\vartheta}{r} = -\frac{1}{r}\frac{\partial p}{\partial \vartheta} + \frac{1}{Re}\left[\nabla^2 v - \frac{v}{r^2 \sin^2\vartheta} + \frac{2}{r}\frac{\partial u}{\partial \vartheta}\right],$$

$$u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \vartheta} + \frac{uw + vw\,ctg\,\vartheta}{r} = \frac{1}{Re}\left[\nabla^2 w - \frac{w}{r^2 \sin^2\vartheta}\right],$$

$$\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \vartheta} + \frac{2u + v\,ctg\,\vartheta}{r} = 0,$$

$$r = 1. \quad u = v = 0, \quad w = \sin\vartheta, \tag{0.2}$$

$$r = 1 + \delta: \quad u = v = 0, \quad w = (1+\delta)\,\omega \sin\vartheta$$

φ, with the corresponding conditions of symmetry for $\vartheta = 0, \pi/2$ or π and three defining dimensionless parameters:

$$Re = \frac{\Omega_1 r_1^2}{\nu}\ ;\ \omega = \frac{\Omega_2}{\Omega_1}\ ;\ \delta = \frac{r_2 - r_1}{r_1} = a - 1 .$$

Here $U = \{u, v, w\}$ are the projections of the velocity on the coordinate axes r, θ, φ ; P is the pressure; ν is the kinematic viscosity coefficient; and $r_1, r_2, \Omega_1, \Omega_2$ are the radii and angular velocities of the inner and outer spheres.

The solution of the boundary value problem in the whole range of the defining parameters can be obtained only with numerical methods. Although the general circulation of planetary /5 atmospheres is basically three-dimensional, the two-dimensional axially symmetric flow can be considered as some approximation for the circulation in the equatorial regions. Moreover, deviations from axial symmetry are often small in comparison with the basic flow and the axially symmetric flow can simulate the averaged meridianal circulation.

The problem of stability of flow arising in a spherical layer, besides being of applied interest, is also of purely hydrodynamic interest. As is well known, a tremendous number of works has been devoted to the study of the stability of plane Poiseuille and Couette flow and analogous flow in tubes and cylindrical layers, and only a few works on the stability of flow in a spherical layer. This is evidently the result of the absence of an analytic solution of the equations for the basic steady-state motion and the significant dependence of this solution on the Reynolds number Re. Both these facts significantly complicate the theoretical investigation of stability. The complexity of fulfilling an experimetal assembly and the rigid limitations on the tolerances have evidently delayed

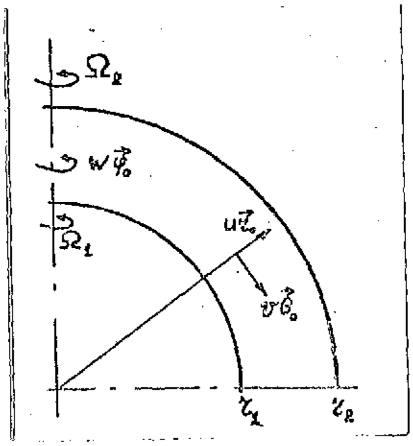

Figure 1. Schematic form of the flow region.

experimental investigations in this direction. However, in spite of all these difficulties, interest in the study of fluid motion in spherical layers has noticeably increased in the last 5 — 6 years and Part 1 of the review presented below reflects the results of the works of the recent years.

Taking into account the effect of thermal effects in stably stratified spherical layers in the Boussinesq approximation and convective flow of fluid under conditions of rotation, shear, etc., are of no less interest for astro- and geophysics. The basic results obtained in this direction over the last decade will be discussed in Part 2 of the review.

SPHERICAL COUETTE FLOW AND ITS STABILITY

§ 1. Basic Steady-State Flow

In a spherical layer with rotation of one or both of its boundaries, there arises fluid flow, which we will call spherical Couette flow by analogy with the plane and cylindrical cases. This flow is mathematically described by the boundary value problem formulated above (0.1) and (0.2), and the nature of this flow is defined by the value of the dimensionless parameters Re, ω, and δ. /6

a) Sufficiently small Reynolds numbers.

For small Re, the solution of the boundary value problem can be obtained analytically by representing the solution in the form of a series in positive integer powers of Re:

$$\overline{U} = \sum_{n=0}^{\infty} \overline{U}_n \, Re^n \qquad (1.1)$$

The convergence of this series for sufficiently small Re was proven in [2]; however, the problem of the radius of convergence remains open. It is clear only that the Reynolds number Re_*, being the radius of convergence of (1.1), depends significantly on the other defining parameters of the problem ω and δ. The zeroth term of the expansion in (1.1) describes the flow when the inertial forces are small, compared to the viscous force. In this classical case, the flow is motion along a circle with a velocity:

$$\overline{U}_0 = [(a^3\omega - 1)r^3 - a^3(\omega - 1)] \sin\theta \, \overline{\varphi}_0 / (a^3 - 1) r^2 . \qquad (1.2)$$

Axially symmetric meridianal flow arises for large Re, which is described by the first term in (1.1) [3, 4, 5]:

$$\bar{U}_1 = \frac{f(r)}{r^2}\left(2-3\sin^2\theta\right)\bar{r}_0 - \frac{f'(r)}{r}\sin\theta\cos\theta\cdot\bar{\theta}_0, \tag{1.3}$$

where

$$f(r) = \frac{(r-1)^2(r-a)^2}{4(a^3-1)^2}\left[Ar + B + C r^{-1} + D r^{-2}\right]. \tag{1.4}$$

The constants A, B, C, and D depend only on a = 1 + δ and ω. /7
Higher terms of the series (1.1) were determined in [6]. However, for solving practical problems, we are mainly interested in the problem of the rate of convergence of the series (1.1), i.e., the accuracy with which n terms of the series approximate the exact solution. Until now, assertions have been encountered in the literature that the expansion (1.1) or the approximate solution (1.2) are valid only for Re $\ll$ 1 [3]. That this is not true can be seen from the results of [4 and 6]. Thus, for thick layers with $\delta \sim 1$ and $\omega = \infty$ (rotation of the outer sphere only), the approximate solution $\bar{U}_0$ + Re $\bar{U}_1$ with Re $\leq$ 10 is practically undistinguishable from the exact [6]. For thin layers $\delta \leq 0.1$ and $\omega \sim 1$, the situation is even better and the ratio $|U_1|/|U_0|$ is sufficiently small [4]; thus, for $\delta \sim 0.1$,

$$|U_1|/|U_0| \sim 10^{-4} - 10^{-5}$$

Thus, in thin layers ($\delta \ll 1$) (and it is these cases which are of greatest interest in astro- and geophysical applications), one can make use of the approximate analytic solution (1.1), taking into account only the first few terms, for sufficiently large Reynolds numbers (~1000 and greater), depending on the specific value of δ.

The motion described by (1.1) can be represented as a differential rotation about the axis with the velocity:

$$\overline{U} = \sum_{i=0}^{\infty} \overline{U}_{2i} Re^{2i}$$

and meridianal flow with velocity

$$\overline{U} = \sum_{i=0}^{\infty} \overline{U}_{2i+1} Re^{2i+1}.$$

The general form of the flow in the meridianal plane for sufficiently small Re is determined by the function $f(\tau)$, whose nature depends to a significant degree on the parameters a and ω. The possible types of meridianal flow depending on the values of the parameters a and ω are presented in Figure 2; I is single vortex circulation with counterclockwise rotation; III is single vortex circulation with clockwise rotation, and II is double vortex circulation with opposite rotations of the vortices. The regions in the a,ω plane, in which one or the other type of circulation occurs, are shown in the same figure. The boundary curves of $\omega_1(a)$ and $\omega_2(a)$ are found from the conditions $\Psi''(a) = 0$ and $\Psi''(1) = 0$, respectively: /8

$$\omega_1 = -\frac{15a^4 + 51a^3 + 77a^2 + 53a + 14}{6a^6 + 42a^5 + 93a^4 + 102a^3 + 56a^2 + 14a + 2},$$

$$\omega_2 = -\frac{2a^6 + 14a^5 + 56a^4 + 102a^3 + 93a^2 + 42a + 6}{(14a^4 + 53a^3 + 77a^2 + 51a + 15)a^2}.$$

The surfaces of equal angular velocities for small Re are close to the concentric spheres.

b) Reynolds numbers $Re \gg 1$ and Almost Rigid Body Rotation $|\omega - 1| \ll 1$

In the other limiting case of very large Re and small $|\omega - 1|$ (the rotational velocities of the spheres are almost the same), an analytic solution is found by the method of inner and outer

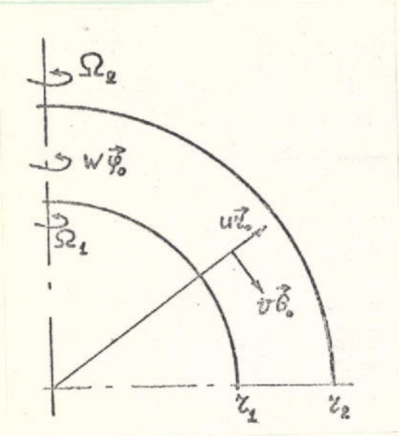

Figure 1. Schematic form of the flow region.

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$$\bar{U}_0 = [(\alpha^3\omega-1)r^3 - \alpha^3(\omega-1)]\sin\theta\,\bar{\varphi}_0/(\alpha^3-1)r^2. \tag{1.2}$$

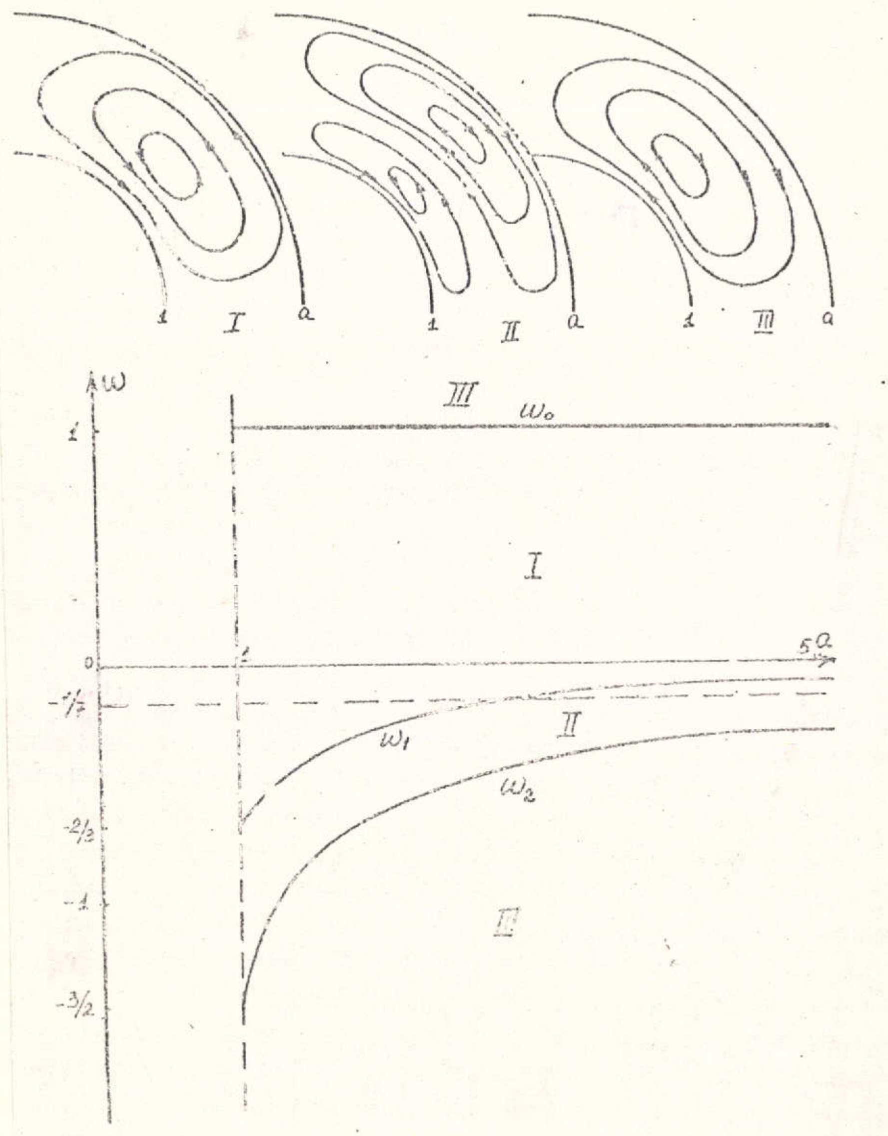

Figure 2. General nature of the meridianal flow and the region of the effect of the spheres, depending on the parameters a and ω for small Re.

asymptotic expansions [7 and 8]. The flow for Re differs not only quantitatively, but also in form from the flow for small Re. It is found that a cylinder of radius R = 1 encompassing the inner sphere with an axis coinciding with the axis of

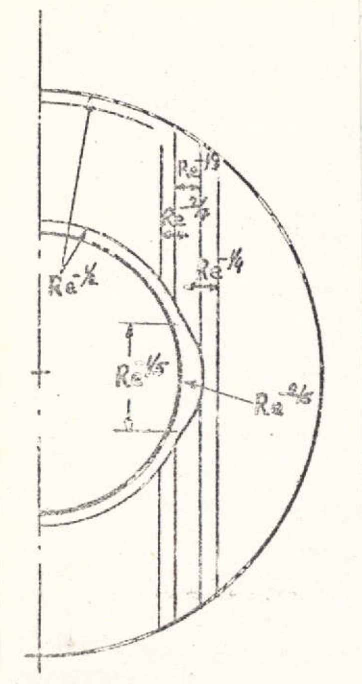

Figure 3. Structure of the Ekman and shear layers for $Re \gg 1, |\omega - 1| \ll 1$.

rotation of the spheres separates regions with different flow characteristics:

$$\Omega_0 = \Omega_2 = const \; ; \quad \Psi_0 \equiv 0 \; , \quad R > 1$$

$$\left.\begin{aligned} \Omega_0 &= (1-R^2)^{\frac{1}{4}}\left[\left(1-\frac{R^2}{a^2}\right)^{\frac{1}{4}} + (1-R^2)^{\frac{1}{4}}\right]^{-1} \\ \Psi_0 &= \frac{R^2}{2\,Re^{1/2}}\left[\left(1-\frac{R^2}{a^2}\right)^{1/4} + (1-R^2)^{1/4}\right]^{-1} \end{aligned}\right\} R < 1$$

Here R is the dimensionless distance from the axis of rotation. Outside the cylinder, the fluid rotates like a rigid body with the angular velocity of rotation of the outer sphere Ω_2. Inside the cylinder in a nonviscous flow core, the angular velocity of rotation and the current function of the meridianal flow depend only on the distance /9
from the axis of rotation. The fluid rotates with an angular velocity intermediate between Ω_1 and Ω_2; the meridianal flow is motion from the slowly rotating sphere to more rapidly rotating cylindrical surfaces with generating lines parallel to the axis of rotation of the spheres. Ekman boundary layers of thickness $Re^{-1/2}$ are formed at the inner and part of the outer spheres, all of the return flow occurs in a thin cylindrical shear layer close to the cylinder of radius R = 1. This shear layer has a very complex structure and actually consists of three separate layers of differing thicknesses, which perform different physical functions (Figure 3). There are two outer layers: one outside the cylinder R = 1 of thickness $Re^{-1/4}$, in which the main return flow of fluid from one Ekman layer to another takes place, and

the second inside the cylinder R = 1 of thickness $Re^{-2/7}$, in which the process of smoothing the discontinuity in the azimuthal velocity occurs. These layers are separated by an inner shear layer of thickness $Re^{-1/3}$, in which the discontinuities in the second derivative of the azimuthal velocity and in the component of the meridianal velocity perpendicular to the axis of rotation are smoothed.

c) Intermediate Reynolds Numbers

Solution of the problem in the case of intermediate Reynolds numbers is the most complex. Obtaining solutions here is possible only by numerical calculation by computer of the nonlinear boundary value problem for the system of partial differential equations (0.1) and (0.2). These solutions have been obtained by difference [9, 10, and 11] and direct [6, 12 — 14] methods. Both types of methods are rather cumbersome, the application of the difference schemes are related to a significant expenditure of machine time; the application of direct methods requires the use of a large number N of basis functions, which leads to algebraic systems of very high orders. Comparison of the results of [6, 9, 10, and 11] (Figures 4 and 5) with the results of [12 — 14] (Figure 6) indicates that, for an insufficient number of basis functions, solution by the Galerkin /10 method can evidently give not only quantitatively, but also qualitatively incorrect results. We note that actual calculations were carried out in [10 and 11], not for an incompressible liquid, but for a monatomic gas with

$$\gamma = c_p/c_v = \frac{5}{3}$$

with a rotational Mach number

$$M_\omega = \frac{\Omega_1 r_1}{c} \approx 0,8$$

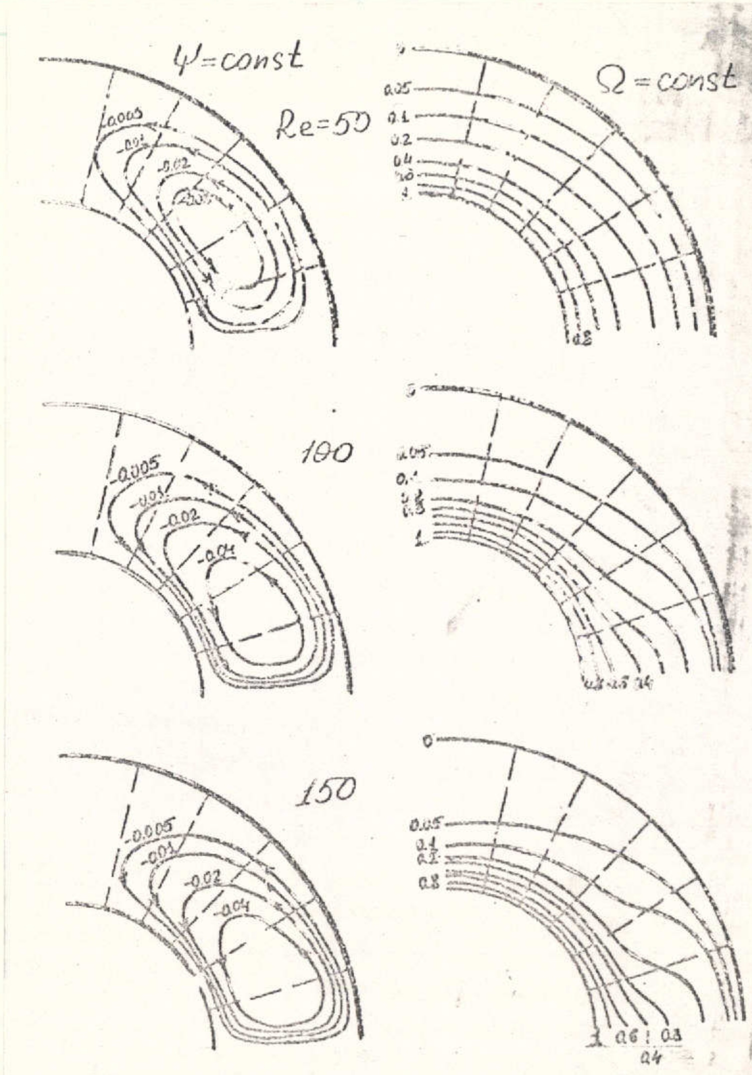

Figure 4. Streamlines and angular velocities for rotation of the inner sphere, a = 2, ω = 0.

((C_p and C_ν are the specific heats of the gas; c is the velocity of sound). However, consideration of the compressibility of the gas in the considered interval of the parameters

$$(50 \le Re \le 500;\ \omega = 0 \div 1;\ \alpha = 2;\ 1.5;\ 1.2;\ 1.1)$$

weakly affected the distribution of velocity, i.e., the Mach number of the meridianal flow was two or three orders smaller

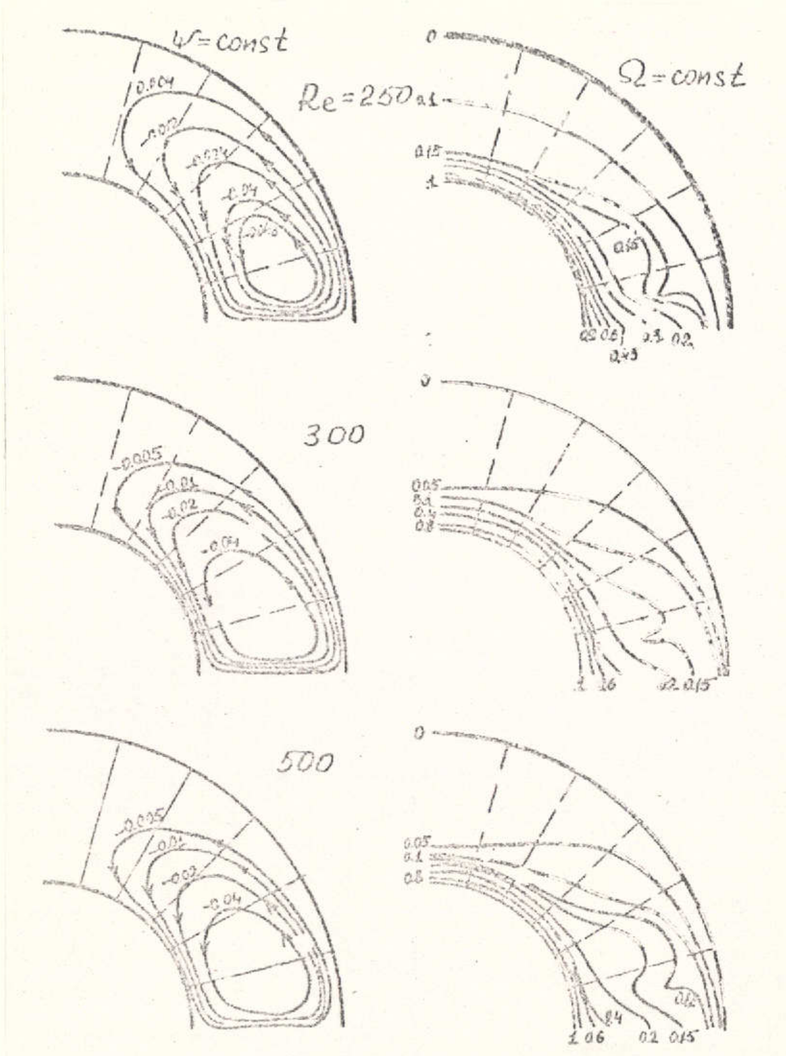

Figure 5. Streamlines and angular velocities for rotation of the inner sphere, a = 2, ω = 0.

than M_{ω}. The results of [10 and 11] agree qualitatively and quantitatively with the results of [6 and 9] obtained for the incompressible liquid.

A certain method intermediate between the difference and direct is proposed in a recently published work [3]. The solutions are expanded in series in powers of the angle θ, the

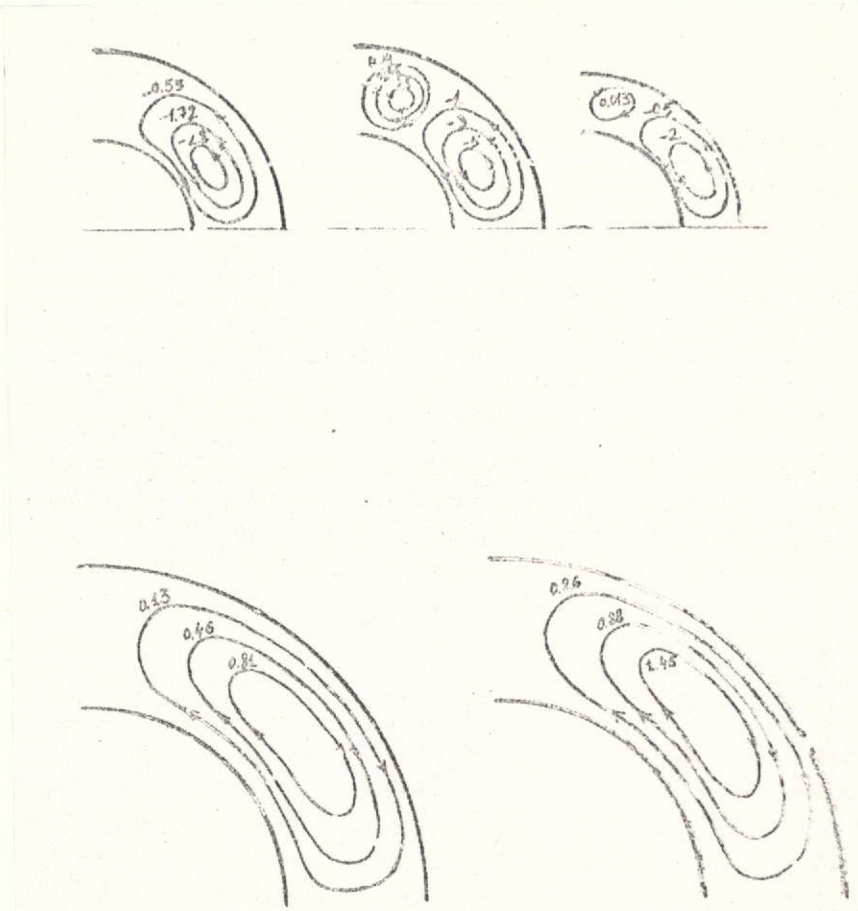

Figure 6. Streamlines for rotation of the inner sphere (top) $\omega = 0$, Re = 59, 119, and 121 (left to right); of the outer sphere (bottom) $\omega = \infty$, $\alpha = 1.7$, $Re_1 = 21.7$ (left to right); $Re_1 \equiv \Omega_2 R_1^2/\nu$.

coefficients in these series are functions of the distance from the center z and an infinite number of coupled systems of ordinary differential equations of fourth order with boundary conditions specified at the two ends of the interval of variation of z is obtained for them. As is usual in the direct methods, the system is cut off at some N and the obtained boundary value problem for N systems of fourth order is solved by the adjustment method. Graphs of the solutions for a thin layer $\delta = 0.1$ in the following intervals of values of Re and ω:

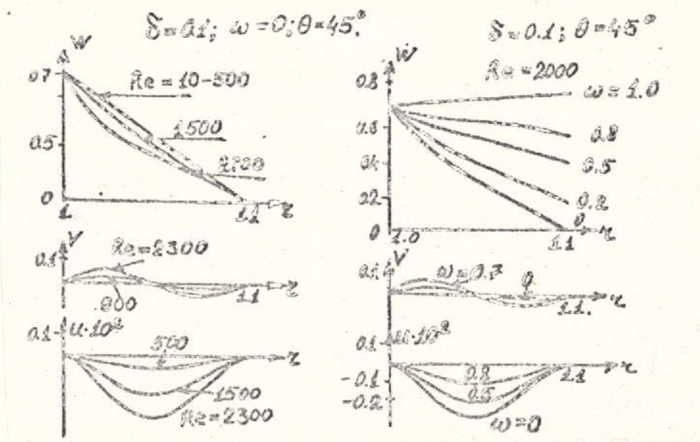

Figure 7. Velocity components depending on Re for ω = 0 (left) and on ω for Re = 2000 (right); δ = 0.1, θ = 45°.

$10 \leq Re \leq 2300$ and $0 \leq \omega \leq 1$

are presented in Figure 7. It is confirmed that the convergence of the method is fairly good in the considered interval of variation of the parameters and N = 5 provides sufficient accuracy of the solution anywhere except the equatorial region.

The results of the works in studying flow in spherical layers, which arises due to rotation of the boundaries with different angular velocities [3 — 14], permits understanding /11 the flow pattern in a wide interval of the defining parameters Re, ω, and δ. As has already been noted, for small Re and arbitrary ω and δ, the basic flow consists of rotation about the axis with a velocity determined by (1.2), on which the meridianal circulation (1.3) is superimposed (Figure 2), whose intensity increases with increasing Re. The calculations indicate that the further development of flow with increasing Re depends significantly of the values of the other parameters: ω and δ.

For $\omega = 0$ (only the inner sphere rotates), the meridianal circulation for Re consists of one vortex with counterclockwise rotation. With increasing Re, the meridianal flow is concentrated in the equatorial region and stagnant zones are formed near the poles (Figures 4 and 5). The surfaces of equal angular velocities are strongly distorted, particularly at the center of the meridianal vortex. The formation of boundary layers is observed on the spheres. The described results are obtained by the difference methods [9 — 11] and the Galerkin method [6] for $\delta = 1$; $Re = 50 \div 500$; $\delta = 0{,}5$; $Re = 100 \div 500$.

For $\omega = \infty$ (only the outer sphere rotates), the meridianal circulation for small Re consists of one vortex with clockwise rotation. With increasing Re, the entire meridianal flow is concentrated in the region inside the cylinder of radius $R = 1$, equalling the radius of the inner sphere with generating lines parallel to the axis of rotation; the region outside the cylinder is almost rigid-body rotation. Regions of large angular and meridianal velocity gradients are formed close to the cylinder $R = 1$ (Figure 8). Boundary layers are formed on the spheres inside the cylinder; in the region of the equator, a weak vortex opposite in direction to the main circulation appears. With increasing Re, a tendency toward the formation of a cylindrical /12 shear layer close to $R = 1$ is seen.

For $\omega < 0$, i.e., when the spheres are rotating in different directions, and for small Re, the type of circulation depends on the values of the parameters a and ω (Figure 2). If, for small Re, the circulation is double vortex ($\delta = 1$, $\omega = 0.5$), then the increase of Re leads to a predominant increase in intensity of the vortex with negative circulation at the inner sphere (Figure 9). If, for small Re, the circulation is single vortex with positive rotation ($\omega = -1$, $\delta = 1$), then an increase

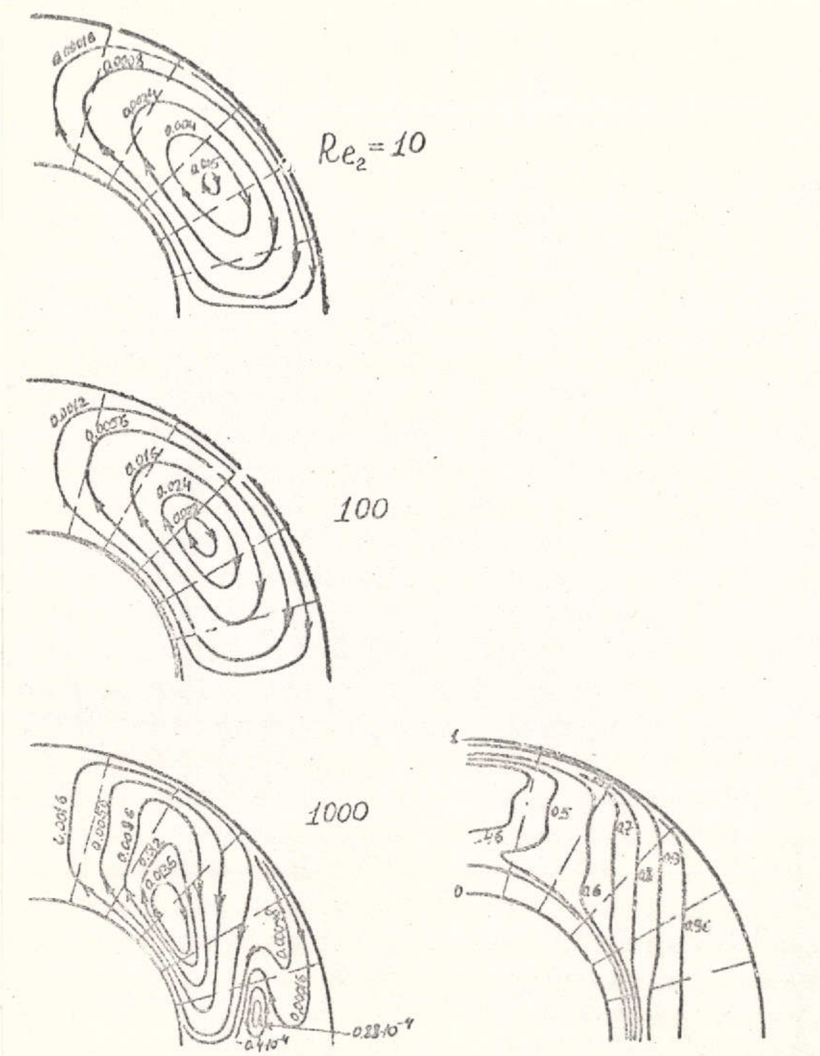

Figure 8. Streamlines and angular velocities for rotation of the outer sphere, ω = 0, a = 2, Re = 10, 100, and 1000 (top to bottom).

of Re leads to the formation of a vortex of opposite circulation at the inner sphere (Figure 10). In both cases (ω = -1; -0.5), there is a tendency with increasing Re toward the formation of a cylindrical shear layer close to the cylinder R = 1.

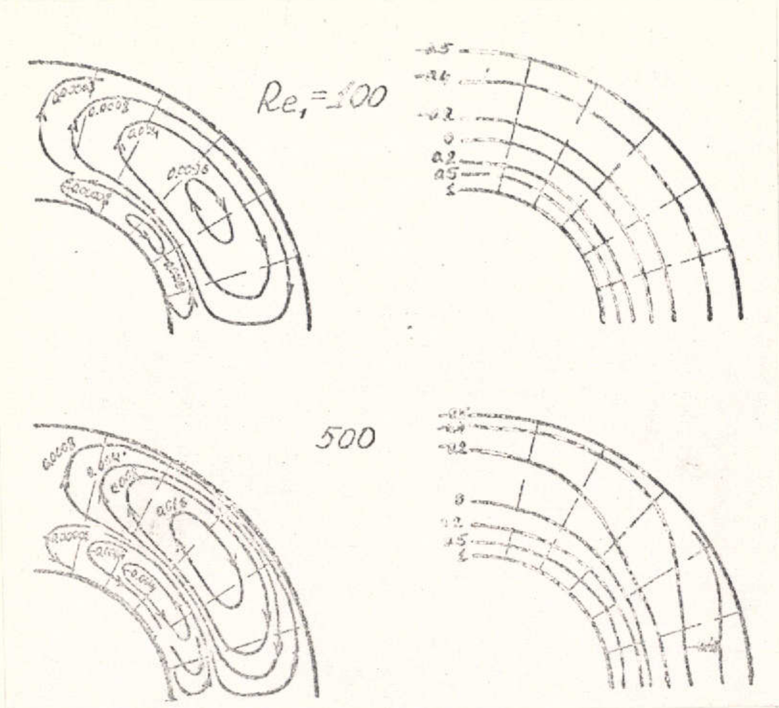

Figure 9. Streamlines and angular velocities when the rotation of the outer sphere is half that of the inner a = 2, $\omega = -0.5$, $Re_1 = 100$, and 500 (top to bottom); $Re_{\omega}^{1} = \Omega_1 R_2^2/\nu$.

The effect of the layer thickness δ on the development of flow for the same values Re = 500, $\omega = 0$ is presented in Figures 11 and 12 (δ = 0.5; 0.2; 0.1). The meridianal flow for all Re consists of one vortex with counterclockwise circulation. With decreasing δ and constant Re, a decrease in intensity of the circulation takes place, the center of the vortex rises and the boundary layers disappear. The surfaces of equal angular velocities in the thin layers (δ = 0.2; 0.1) are almost concentric spheres. The radial distribution of velocities at an angle $\theta = 45°$ for various ω in the interval (0; -1) and constant Re = 2000 and for fixed $\omega = 0$ and various Re from 500 to 2700 is presented in Figure 7. Comparison of the results presented in Figures 4, 7, 11, and 12 indicates that there is qualitative agreement of flow in thin layers for large

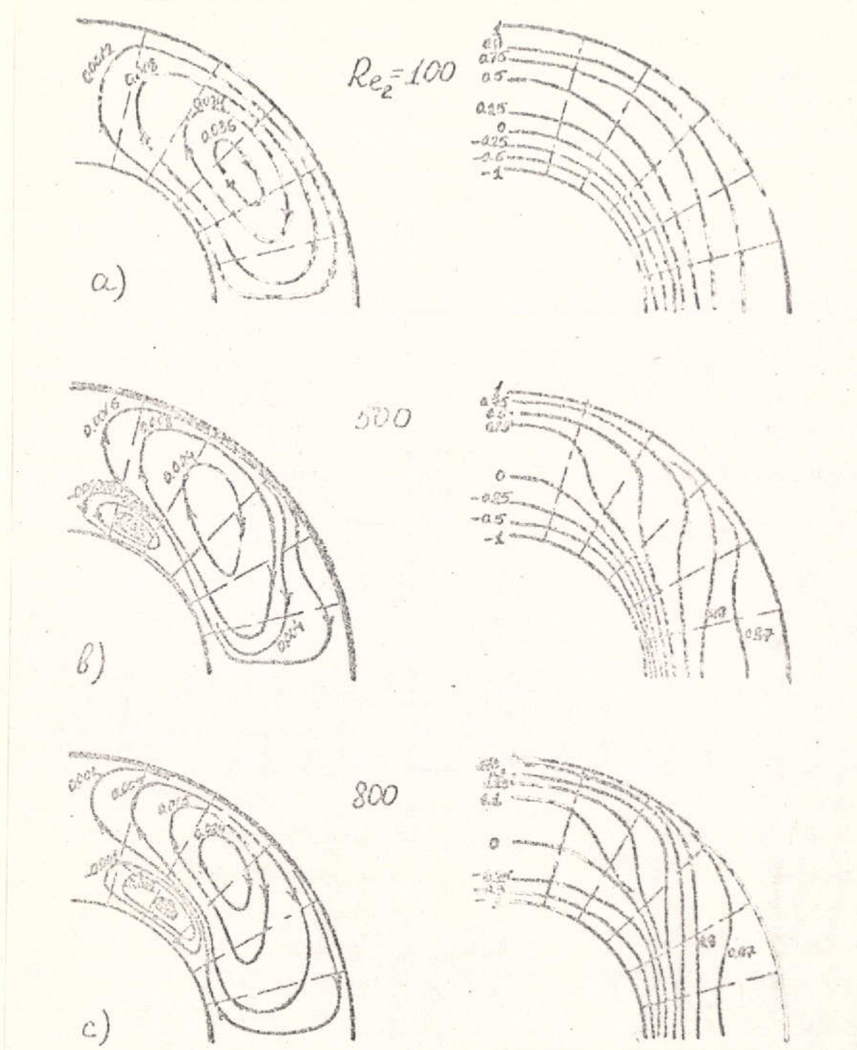

Figure 10. Streamlines and angular velocities for rotation of both spheres with the same angular velocities in opposite directions, a = 2, $\omega = -1$, $Re_2 = 100$, 500 and 800 (top to bottom); $Re_2 = \Omega_2 R_2^2/\nu$.

Re and in thick for small Re. However, it is impossible to find one combination of dimensionless parameters Re and δ, which would provide similarity of flow for any values of these parameters.

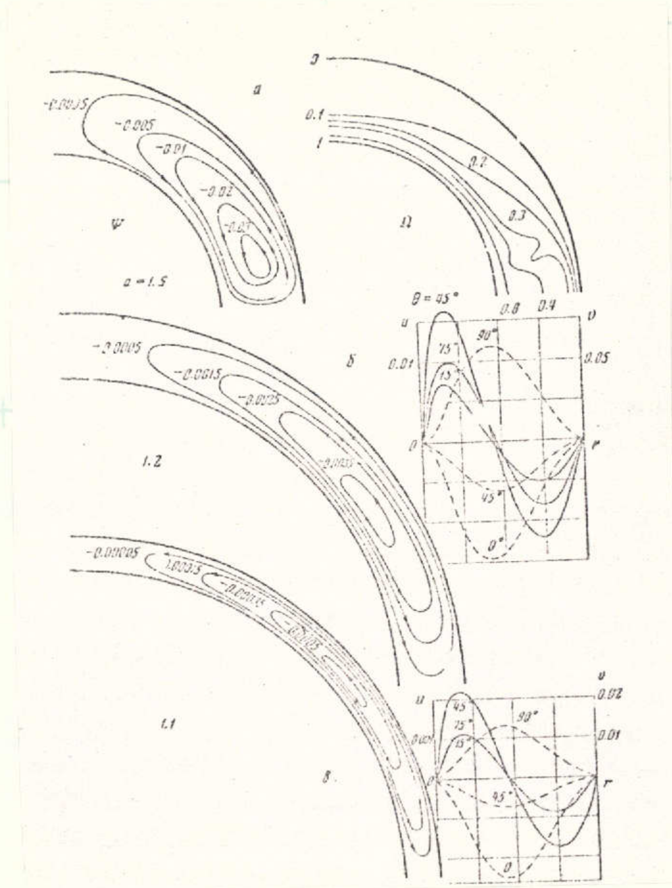

Figure 11. Streamlines, components of meridianal velocity and angular velocities for rotation of the inner sphere only for various thicknesses of the fluid layer, Re = 500, a = 2, 1.5, and 1.1 (top to bottom).

§ 2. Stability of Spherical Couette Flow /13

It is known that, for large Reynolds numbers, the established laminar flow becomes unstable and, consequently, physically unrealizable. This problem arises with the study of flow in a spherical layer. Besides the natural necessity of identifying the critical Reynolds numbers, the study of stability of the flow under consideration also has purely hydrodynamic interest for two reasons:

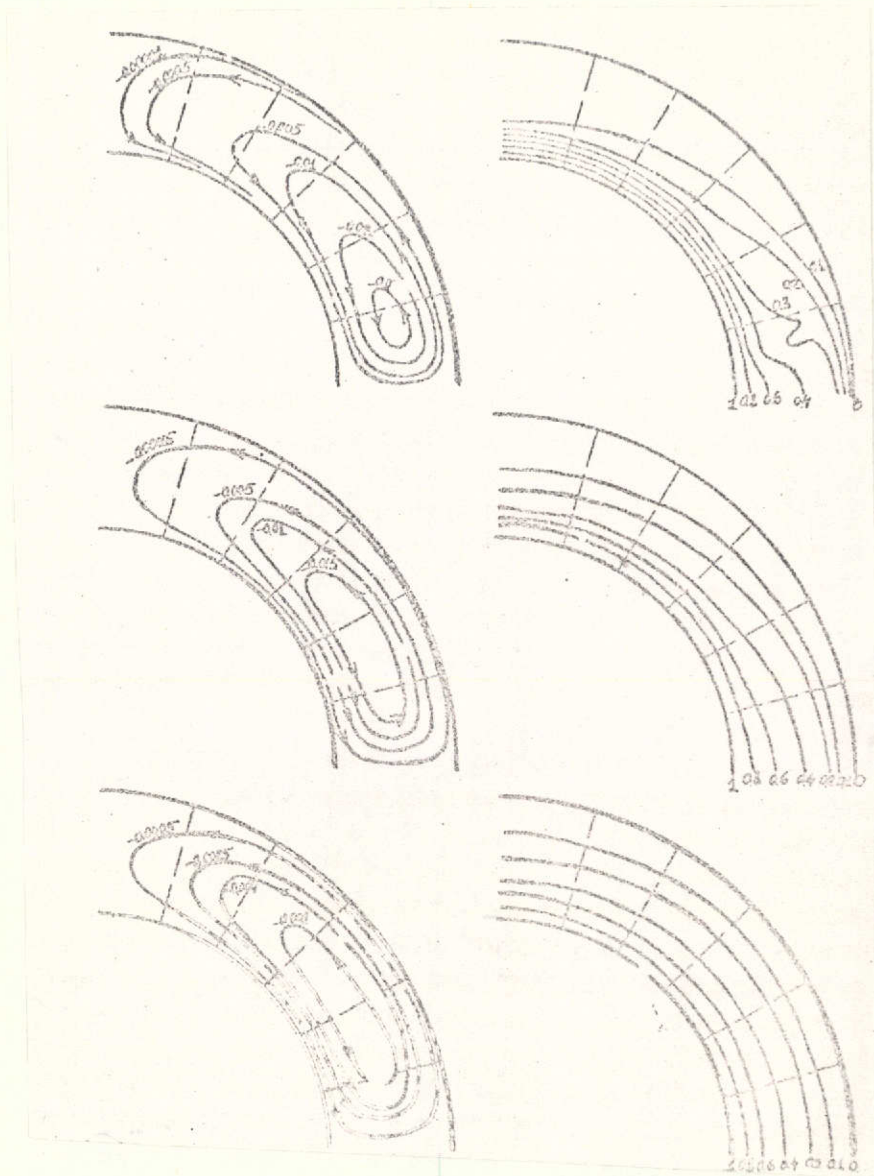

Figure 12. Streamlines and angular velocities for rotation of inner sphere, ω = 0, a = 1.5, Re = 100, 200, and 500 (top to bottom.

1. The investigated flow occurs in a closed region, in contrast to the usually considered plane and cylindrical Couette and Poiseuille flow;

2. The nature of the basic flow depends on Re, which is also unusual for this type of problem.

Results of investigations of the stability of flow in spherical layers is very sparse, in view of the great complexity of the problem [15 — 19]. The stability of flow relative to infinitesimal axially symmetric perturbations was investigated in the works of V. I. Yakushin [15 — 18] and the stability relative to finite non-axially symmetric perturbations was investigated by the energy method in the work of Munson and Joseph [19]. A summary of the results of [15 — 19] is presented in Table 1. However, as is seen in the table, the results of these works do not permit obtaining a unified physical picture of the occurrence of instability in spherical layers. Actually, in [15 and 16], the stability was studied for flow in thin layers relative to infinitesimal axially symmetric perturbations for rotation of the inner sphere only ($\omega = 0$) or for rotation of both spheres in the same direction ($\omega > 0$). The critical Reynolds numbers were found and curves were obtained analogous to the Taylor curves for the rotation of cylinders and representing the boundary of stability in the planes

$$Re, Re_{12} = \Omega_2 r_1^2 / \nu$$

for $\omega \geq 0$; $\delta = 0.07$ and $\delta = 0.1$ (Figure 13). The fluid motion /14
in thin layers $\delta = 1$ and 0.7 relative to the same perturbations for the rotation of the inner sphere only ($\omega = 0$) is always stable [17]. Flow with rotation of the outer sphere ($\omega = \infty$) and layer thicknesses of $\delta = 0.07$ and $\delta = 0.7$ is also stable [18]. The results of [15 — 18] give the justification to assume that the parameters ω and δ play the deciding role in the stability of flow in a spherical layer relative to infinitesimal axially symmetric perturbations. The conclusions follow from [15 — 18]:

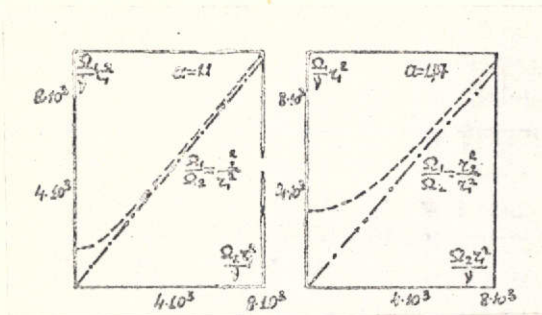

Figure 13. Curves of stability of motion in a thin spherical layer, a = 1.1 and 1.07 (left to right).

1. Flow in layers of arbitrary thickness for rotation of the outer sphere only ($\omega = \infty$) is always stable;

2. In thin layers ($\delta < \delta^*$) for sufficiently large $Re \geq Re_{cr}(\delta)$ and $\omega \geq 0$, flow becomes unstable. However, the thinner the layer, the more stable it is, i.e., the larger $Re_{cr}(\delta)$. The value of δ^* is not established in the works; however, there is experimental evidence that $\delta^* \approx 0.19$ for $\omega = 0$ (cf., below).

3. Flow in sufficiently thick layers $\delta > \delta^*$ is always stable, independent of the values of Re and ω.

4. Perturbations causing the onset of instability at the stability limit are a system of annular vortices, whose intensity decreases from the equator towards the pole; the circulation in neighboring vortices is in the opposite direction.

In contrast to the works of the cycle [15 — 18], the stability of flow for rotation of both spheres in different directions ($\omega < 0$) was also investigated [19]. As is well known, the energy method used in [19] makes it possible to obtain only the lower limit of stability, i.e., the value of Re^E, below which the flow is always stable. The results of [19] permit reaching the following conclusions about the stability of flow in spherical layers relative to finite perturbations (cf., Table 1).

TABLE 1

STABILITY OF FLOW IN SPHERICAL LAYERS

	ω	δ	Critical values of Reynolds number	Conclusions	Article
Method of small perturbations	0	0.07	Re* = 3260	Instability relative to axially symmetric perturbations	Yakushin, V. I. MZhG, 1969, [illegible] No. 1
	0	0.1	Re* = 1600		
	$1 < \omega \leq 0$	0.07	Re* = f(1.07, ω)	Form of the stability curve relative to axially symmetric perturbations were obtained for δ = 0.1; 0.07.	Yakushin, V. I. MZhG, 1970, No. 2
	$1 < \omega \leq 0$	0.1	Re* = (1.1, ω)		
	[illegible] < ∞	0.07	Flow is stable relative to axially symmetric perturbations in the specified interval	$0 \leq Re \leq 300$	Yakushin, V. I. Uch. Zap. Perm. Univ., 1970, No. 216
	∞	0.7		$0 \leq Re \leq 71$	
	0	0.7		$0 \leq Re \leq 190$	Yakushin, V. I. Uch. Zap. Perm. Univ., 1971, No. 248
	0	1		$0 \leq Re \leq 120$	
Energy method	∞	1	Re_2^{*} = 110	Nonaxially symmetric perturbations with an azimuthal wave number m = 1 are most disruptive	Munson, B. R., Joseph. D. D. J. Fl. Mech., 1971, Vol. 49, No. 2
	-1	1	Re^E = 47.5		
	-0.5	1	Re^E = 55		
	0	1	Re^E = 90		
	∞	0.7333	Re_2^E = 190		
	0	0.333	Re^E = 172.5		

* $Re_2 = \Omega_2 R_2^2/\nu$.

1. Motions arising with the rotation of the outer sphere only ($\omega = \infty$); /15

2. Motions in thinner layers are more stable;

3. The most disruptive perturbations for all the considered ω and δ are nonaxially symmetric perturbations with an aximuthal wave number $m = 1$, which are characterized by a vertical flow through the equatorial plane, horizontal flow through the pole, and two vortices of complex structure and opposite directions close to the equator, Re^E is a function of ω and δ.

A direct comparison of the critical values Re^* and Re^E obtained by these two methods is clearly pointless. Nonetheless, the first and second conclusions reached from both methods agree qualitatively, i.e., the flow is more stable in thinner layers and with the rotation of the outer sphere only. However, such a decisive role of the parameters ω and δ on the flow stability does not follow from the energy method, as in the method of small perturbations; that is, a change in the values of ω and δ in [19] changes only the critical value of the Reynolds number Re^E, whereas the general conclusion about the stability remains unchanged: for all considered values of ω and δ (cf., Table 1) the most disruptive are the same nonaxially symmetric perturbations with $m = 1$ (we note for accuracy that the stability of flow in thin layers with $\delta < \delta^*$ was not investigated with the energy method).

Thus, the results obtained in [15 — 19] cannot always agree between themselves, although at this stage of the investigations, it is too early to indicate any contradictions. Analogies with Couette flow between cylinders [20 and 21] and convective instability in a spherical layer [22 — 25] are of little help for a qualitative understanding of the instability phenomenon. In these problems, related to some extent, the layer thickness δ does not play such a decisive role in the problem /16 of flow stability, i.e., in these problems, there is no such critical values of layer thickness δ^*, above which ($\delta > \delta^*$) the flow would remain stable relative to infinitesimal perturbations for any values of the other parameters. The parameter ω, as was explained, was found to be a more significant parameter in the cylindrical Couette flow. For sufficiently large negative ω ($\omega < -0.76$), i.e., for rotation of the cylinders in the opposite directions, the loss of stability of the main flow beings not relative to axially symmetric perturbations, but relative to non-steady-state nonaxially symmetric perturbations with an azimuthal wave number m, depending on ω [21]. A spherical layer with slow rotation is also convectively unstable relative to similar perturbations [22, 25], only here the azimuthal wave number m equals the poloidal wave number l; the value $m = l$ is determined by the thickness of the spherical layer δ. It should be emphasized that, in these problems [21, 22, and 25], the nonaxially symmetric perturbations causing the instability, in contrast to the results of [15], are not steady-state. Thus, if the Galerkin method in [15 — 18] gave correct results on the instability of motions in a spherical layer relative to infinitesimal perturbations and the energy method [19] permitted finding the critical Reynolds numbers close to the critical Reynolds numbers Re of the linear problem, then the picture of flow instability arising in a spherical layer with shear is very unusual and dissimilar to all investigated analogous problems. Further investigations in this direction are necessary and, in particular, consideration of the instability of flow relative to nonaxially symmetric perturbations in the linear problem and obtaining solutions of the problem in a wider range of values of the parameters ω and δ to formulate the complete picture of flow in spherical layers. Much confusion of the theory could be clarified experimentally. However, we know of only two works [26 and 27] in which flow in spherical /17 layers was studied. Both works were performed on similar apparatus at the Perm State University.

The divided outer sphere was cut from plastic and made transparent for visualization of the observations. Its radius $r_2 = 3.244$ cm remained fixed. The interchangeable inner spheres were also cut from plastic. Distilled water and aqueous solutions of glycerine were used as the working liquid. Aluminum powder and various types of dyes were used for visualization of the flow. Motions arising in the liquid with rotation of the inner sphere only $\omega = 0$ and layer thicknesses $\delta = 0.0371$; 0.0745; 0.1225; 0.1901; 0.4443; and 1.5147 were studied. The torque M acting on the outer sphere* was measured and visual observations of the flow characteristics were carried out for various layer thicknesses and angular velocities for Re. The results of the experiment are presented in Figure 14 and Table 2. It is obvious that, if the solution is represented in the form of a sum of three terms of the series (1.1), then:

$$w(r, \theta) = w_0(r)\sin\theta + Re^2 w_2(r) P_2(\theta)\sin\theta,$$

and the dimensionless torque is defined as:

$$\mu = M(a^3 - 1)/8\pi a^3 r_1 \eta \nu (1 - \omega) = Re\,[1 + c\,Re^2], \qquad (2.1)$$

where the constant c is defined by the form of the function $w_2(r)$

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*$M = 2\pi \int_0^\pi \tau_{r\varphi} r_2^3 \sin^2\theta \, d\theta$, where $\tau_{r\varphi} = \eta\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)$ are the stresses acting per unit area perpendicular to the radius in the φ direction, η is the dynamic viscosity coefficient.

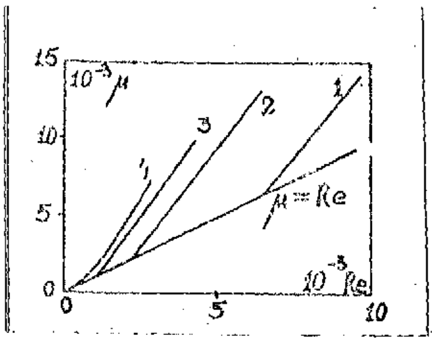

Figure 14a. Experimental dependence of μ on Re for various thicknesses of the fluid layer, ω = 0.

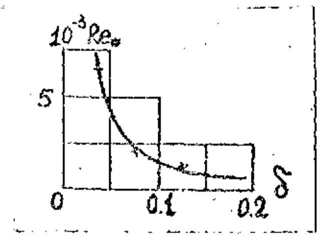

Figure 14b. Experimental dependence of critical Re on thickness of the fluid layer, ω = 0.

For slow rotation, when the solution is described with a high degree of accuracy by the first two terms in the expansion (1.1), the torque μ is a linear function of the Reynolds number. As is seen in Figure 14, in accordance with theory (2.1), the experiment gives the linear dependence μ = Re for all investigated values of δ for small Re. The experiment confirms, in agreement with the theory, the significant dependence of the motion on the layer thickness δ with increasing Re.

a. For thick fluid layers δ > 0.19 (ω = 0), no stability loss is observed with increasing Re. The interval of Re, in which the linear law occurs, is determined by the layer thickness δ; thus, for δ = 0.444, 0 < Re ≤ 100, and for δ = 1.515, 0 < Re ≤ 20. Deviation from the linear law begins with further increase of Re and, after some intermediate region, the dependence $\mu \sim Re^{3/2}$ is established, which is characteristic of the boundary layer. The visual observations indicate that the fluid motion agrees well qualitatively with the motions found theoretically in [4, 9, and 10] and differs from [12]. It consists of rotation and a meridianal flow directed from the poles toward the equator near the outer sphere and in the

TABLE 2*

(from [26])

Number	δ	Re^*	k
I	0,037I	6500 ± 700	2,47 ± 0,06
2	0,0745	2300 ± 200	2,63 ± 0,07
3	0,I225	I250 ± 90	2,88 ± 0,07
4	0,I90I	540 ± 20	-
5	0,4443	-	-
6	I,5I47	-	-

*Commas in the numbers indicate decimal points.

opposite direction at the inner. This motion is symmetric with respect to the plane of the equator. The meridianal flow changes qualitatively with increasing Re: the flow lines are deformed, being increasingly compressed toward the boundaries of the region. A boundary layer is formed, adjacent to the spheres and closed at the poles and along the plane of the equator, which divides the liquid into two symmetric flows [16]. As is evident from the description of the flow, separation of the flow into two meridianal vortices, as obtained in [12], does not occur.

b. In the case of thin fluid layers $\delta \leq 0.12$, an increase of Re leads to the loss of stability of the basic motion, which causes the appearance of the discontinuity in the graph (Figure 14a) for some Re, depending on the layer thickness. With a further increase of Re, a new steady-state motion is established, which is characterized by another linear dependence:

$$\mu = k\left(Re - Re^* \frac{k-1}{k}\right). \qquad (2.2)$$

As is seen in Table 1, the angular coefficient k tends almost linearly on the layer thickness δ. It is observed visually that the loss of stability of the basic flow is characterized by the formation of "annular vortices in the equatorial belt," which remain stable and stationary in the entire region over which Relation (2.2) is satisfied. They are qualitatively similar to the Taylor vortices between rotating coaxial cylinders [26]. The structure of the vortices and the velocity distribution in them and in the remaining part of the liquid were not investigated. It is seen in Table 2 that the thinner the fluid layer, the more stable its motion. An empirical dependence of the critical Reynolds number on the fluid layer thickness is found by analogy with the Taylor flow (Figure 14b [26]): /19

$$Re^* \delta^{3/4} = 49.0.$$

c. In the intermediate region $0.12 \leq \delta \leq 0.19$, loss of stability and the formation of vortices are observed; however, the dependence (2.2) takes place only in a very small interval of the Reynolds numbers. The data of the unique experiments described in [26 and 27] are very interesting and productive for the study of flow in spherical layers, but extremely insufficient in the range of variation of the parameters ω and δ and in the technique for visualizing the flow.

Conclusion

The survey of [3 — 20] permits reaching some general conclusions on the flow in spherical layers.

I. For small Reynolds numbers, the general nature of the generated flow is represented in Figure 2 and is described by the first terms of the series (1.1). The concept itself of the smallness of Re depends significantly on the values of the other parameters ω and δ. Thus, in thin layers for $\delta = 0.1$, /20 Re ~ 1000 can be considered small and Formula (1.2) can be used to describe the flow with a high degree of accuracy.

II. The development of flow with increasing Re depends significantly on the parameters ω and δ. The asymptotic solution for $Re \to \infty$ exists only for the almost rigid body rotation $|\omega-1| \ll 1$ under the condition that $(\omega-1) Re^{1/3} \ll 1$. In this case, the motion outside a cylinder of radius R = 1 does not depend on the sign of $\omega - 1$ and reduces to rigid body rotation with the angular velocity of rotation of the outer sphere. Inside the cylinder, the velocity of the meridianal flow is parallel to the axis of rotation and is directed from the slowly toward the rapidly rotating sphere, the angular velocity of the flow depends only on the distance from the axis. A "suspended" shear layer with large velocity gradients is formed close to the cylinder R = 1. Numerical calculations carried out in [9] for $\delta = 1$; $\omega = 1.0526$ (Figure 15) indicate an approach toward the obtained asymptotic behavior with increasing Re. For finite $\omega - 1$, the sign of $\omega - 1$ is significant. For sufficiently large velocities of the rotation of the outer sphere $\omega = \infty$; -1; -0.5, the formation of the Stewart shear layers and an approach toward the asymptotic behavior described above are observed with increasing Re (Figures 8, 9, and 10). With rotation of the outer sphere only $\omega = 0$, the asymptotic behavior for $Re \to \infty$ must be significantly different: calculations [10 and 11] indicate that the meridianal flow is concentrated in the equatorial zone and zones of stagnation are formed near the poles with increasing Re.

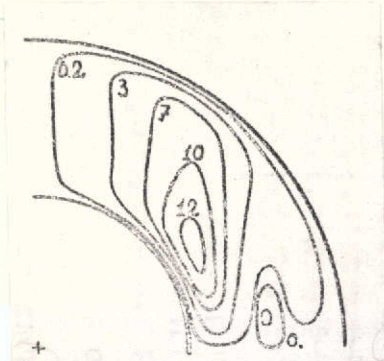

Figure 15. Patterns of streamlines (10^5 Ψ) for ω = 1.0526, Re_2 = 1000, δ = 1.

III. The layer thickness plays the deciding role in the development of flow instability (method of small perturbations) with increasing Re [15 — 19]. For $\delta > \delta^*$ ($\delta^* \sim 0.19$ according to experiment [27]), flow in spherical layers is always stable relative to infinitesimal axially symmetric perturbations [19]. In thin layers $\delta < \delta^*$ and $1 < \omega \leq 0$, the flow can be unstable and a boundary of stability in the Re, Re_{12} plane is found [13], analogous to the Taylor boundary /21 in the flow between cylinders. Thin layers are stable relative to small axially symmetric perturbations for rotation of the outer sphere only. The experiments [26, 27] carried out for the case of rotation of the inner sphere only and various δ give results in qualitative agreement with the theory.

IV. The theory of flow stability [19] relative to arbitrary finite perturbations (the energy method) indicates that the most disruptive perturbations are nonaxially symmetric perturbations with an azimuthal wave number m = 1.

Thus, the available results on the study of spherical Couette flow indicate the necessity of continuing investigations in the following directions:

a. a theoretical investigation of the asymptotic behavior of the basic flow in thick layers for Re → ∞ and its significant dependence on the values of the parameters ω and δ;

b. an investigation of flow stability relative to infinitesimal three-dimensional perturbations; there is particular interest here in obtaining the stability boundary for the layer thickness $\delta = \delta^*$ and, in investigating flow stability for large negative ω;

c. performance of experimental investigations over a wider range of the parameters and with more refined measuring and visualization techniques.

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